Waveform Design for Compressive Radar Sensing

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***Abstract*—Wideband multi-frequency chirp waveforms combined with stretch processing on receive provides a way to obtain linear projection of range profiles at sub-Nyquist sampling rates. Stable recovery of high resolution range profiles from these projections is guaranteed only if the mutual coherence between the columns of the sensing matrix is sufficiently small. In this note, we derive the sensing matrix for the compressive radar sensor with multi-frequency chirp waveforms and analyze its coherence structure. We show that for suitable choice of system parameters the inter-column coherence of unstructured random sensing matrices is achieved.**

I. INTRODUCTION

Imaging sensors achieve high resolution using wideband modulated waveforms and match filtering on receive. Digital implementation of match filtering requires sampling of the received waveforms at the Nyquist rate of the transmit waveform. For the special case of chirp waveforms with linear frequency modulation (LFM) the match filtering can be implemented through a combination of analog mixing stage and sampling at a fraction of the Nyquist rate proportional to the ratio of the target delay support to the pulse width. The resolution of the LFM waveforms is limited by the total bandwidth of the transmitted waveform. Even for sparse scenes with few nonempty range bins, reconstruction beyond this resolution is not guaranteed due to high coherence between returns from adjacent range bins. In the following we derive the sensing matrix for stretch processed LFM waveforms and note that it results in highly correlated columns when range dimension is oversampled. Then we show that the coherence between columns can be minimized when multi-frequency chirp waveforms are used in illumination.

A. Sensing Matrices for Radar Imaging

We consider the following system model for radar sensing. The radar sensor transmits a waveform $\boldsymbol{\phi}(t)$ which is convolved by the target channel response $\boldsymbol{h}(t)$ and then filtered by the receive filter $r(t)$ and sampled. We assume the waveform $\boldsymbol{\phi}(t)$ is the complex baseband form of the transmitted waveform with finite energy E. The noise waveform $n(t)$ is a Gaussian random process with known constant power density, bandlimited to transmitter bandwidth.

Discretization of the sensing model results in the vector model given in Equation (1), where the $M \times 1$ measurement vector $\boldsymbol{y}$ is a noise-corrupted version of the transmitted waveform $\boldsymbol{\phi}$ that has propagated through a sequence of two linear operators: target channel filter $\boldsymbol{H}$ and receiver shaping filter $\boldsymbol{R}$.

$$\boldsymbol{y} \;=\; \boldsymbol{R}\boldsymbol{H}\boldsymbol{\phi} + \boldsymbol{n} \qquad (1)$$

The convolution matrix $\boldsymbol{H}$ is formed using the unknown target response, whereas the receive filter convolution matrix $\boldsymbol{R}$ is formed using the known filter impulse response $r(t)$. The vector $\boldsymbol{n}$ represents sensor output noise with complex circularly symmetric additive white Gaussian noise with known variance σ_n. Linearity of the channel operator $\boldsymbol{H}$ enables us to write the channel output vector $\boldsymbol{H}\boldsymbol{\phi}$ equivalently as $\boldsymbol{\Phi}\boldsymbol{h}$, where $\boldsymbol{h}$ is an $N \times 1$ vector of channel impulse response $h(t)$, and $\boldsymbol{\Phi}$ is a linear operator appropriately constructed from $\boldsymbol{\phi}$. This results in $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{\Phi}\boldsymbol{h} + \boldsymbol{n}$. In addition we consider target responses with a sparse representation in some given basis $\boldsymbol{\Psi}$ (i.e., $\boldsymbol{h} = \boldsymbol{\Psi}\boldsymbol{x}$ for sparse $\boldsymbol{x}$), resulting in the standard sparse sensing model:

$$\boldsymbol{y} \;=\; \boldsymbol{A}(\phi, r)\boldsymbol{x} + \boldsymbol{n} \qquad (2)$$

where the $M \times N$ matrix $\boldsymbol{A}(\phi, r) \triangleq \boldsymbol{R}\boldsymbol{\Phi}(\boldsymbol{\phi})\boldsymbol{\Psi}$ serves as the sensing matrix of the radar system.

B. Sparse Signal Recovery and Mutual Coherence

Compressed sensing research considers the linear inverse problem given in Equation (2), which is recovery of a signal $\boldsymbol{x}$ from measurements of its (noisy) linear projections. The emphasis is on the underdetermined problem where the sensing matrix $\boldsymbol{A}$ forms a non-complete basis with $M < N$. The resulting ill-posed inverse problem can be regularized if the unknown

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signal $\boldsymbol{x}$ is known to have at most K non-zero entries. Results in CS theory provides sufficient conditions for stable inversion of the forward problem given in (2) for appropriate forward operators $\boldsymbol{A}$. In particular conservative sufficient conditions for recovery can be formulated through the *mutual coherence* of the columns of A defined as $\mu(\boldsymbol{A}) = \max_{i \neq j} \frac{|\boldsymbol{A}_i^H \boldsymbol{A}_j|}{\|\boldsymbol{A}_i\| \|\boldsymbol{A}_j\|}$. Well known results [1], [4], [5] links mutual coherence to recovery guarantees. In particular, if:

$$K < \frac{1}{2}\left(1 + \frac{1}{1 + \mu(\boldsymbol{A})}\right) \tag{3}$$

then for each measurement vector y_0 there exists a unique signal x_0, such that $y_0 = Ax_0$. Under this condition, Basis Pursuit and Orthogonal Matching Pursuit algorithms are guaranteed to recover the signal in the absence of noise. Furthermore, the mutual coherence provides an upper bound on RIP-constant δ_R through $\delta_R < (K-1)\mu$. RIP constant quantifies how close the subset of columns of $\boldsymbol{A}$ are close to being isometry and a low RIP constant guarantees stable recovery under noise.

Practically all radar sensors can be formulated as linear operators acting on range profiles. However stable recovery at high resolution is guaranteed only if the mutual coherence of the resulting sensing operator is sufficiently small. In the next section we analyze the mutual coherence of a compressive radar sensor employing multi-frequency chirp waveforms.

II. Sensing matrices for single and multi-frequency chirp waveforms

A. Single chirp transmit waveform with stretch processing receiver

First we consider a radar sensor employing a single chirp transmit waveform $\phi(t) = e^{j\pi\beta t^2/\tau}$ sweeping a total bandwidth of β Hz, over τ seconds expressed in complex baseband notation. The stretch receiver uses a mixing waveform $e^{-j\pi\beta t^2/\tau}$ and taking N samples at a sampling rate of $\Delta T = \tau/(T_u\beta)$ over the sampling interval of τ, where T_u denotes the unambiguous range interval in seconds and $N = T_u\beta$. The echo from a point single target at time delay δ with complex reflectivity $x(\delta)$ results the following discrete time output vector sampled at the output of the stretch receiver

$$y[k] = e^{-\frac{j2\pi\beta\delta}{\tau}(k\Delta_T)} e^{j\pi\beta/\tau\delta^2} x(\delta) \tag{4}$$

$$= e^{-\frac{j2\pi\delta}{T_u}k} e^{j\pi\beta/\tau\delta^2} x(\delta) \tag{5}$$

Inspection of the first term reveals that the return from the target is a complex sinusoid with normalized frequency $2\pi\delta/T_u$. The second term is a complex valued constant known for a given value of δ . In SAR literature it is termed as the residual video phase and it can be easily compensated for after estimation of the target amplitudes.

We focus on the underdetermined problem of reconstruction where the range interval T_u is oversampled by a factor of c where the element of the unknown reflectivity vector $x[l]$ represent the complex reflectivity at delay $\delta = i\frac{T_u}{cN} = \frac{i}{c\beta}$. Therefore we have:

$$y_i[k] = e^{-j2\pi i/(cN)k}$$

The sensing matrix A for a single chirp with the range oversampled by a factor c is simply the oversampled DFT matrix $A = [A_1 \ A_2 \ \cdots \ A_{cN}]$ with column i given by $A_i = \begin{bmatrix} 1 & e^{-j2\pi i/(cN)} & \dots & e^{-j2\pi i(N-1)/(cN)} \end{bmatrix}^T$ if a single chirp waveform $\phi(t) = e^{j\pi\beta t^2/\tau}$ with a bandwidth β and pulse length τ is used to illuminate the target. The resulting coherence between the columns can be expressed in terms of the Dirichlet kernel $D_N(x)$

$$\mu_{i,j} = \frac{D_N(2\pi|i-j|/cN)}{N} \tag{6}$$

where

$$D_N(x) = \sum_{k=0}^{N-1} e^{jkx} = e^{j\omega(N-1)/2} \frac{\sin Nx/2}{\sin x/2} \tag{7}$$

For fixed β increasing oversampling factor c results in large coherence as $D_N(2\pi/cN)/N$ converges to unity as c increases. Figure 1 shows the intercolumn coherence of the sensing matrix A for a traditional radar sensor with single chirp on transmit and stretch processing on receive.

B. Multi-frequency chirp transmit waveform with stretch processing receiver

In this section we will consider a transmit waveform which is the sum of frequency shifted versions of the chirp used in the stretch receiver. We start with a single frequency shifted chirp of the form

$$\phi(t, f_s) = e^{j2\pi\left(f_s t + \beta\frac{t^2}{2\tau}\right)} \tag{8}$$

where f_s is chosen from the interval $[0, c\beta]$. The echo from a point single target at time delay δ results the following discrete time output vector sampled at the output of the stretch receiver (ignoring the common residual video phase term):

$$\begin{aligned} y[k] &= e^{j2\pi f_s(k\Delta_T)} e^{-j2\pi f_s\delta} e^{-\frac{j2\pi\delta}{T_u}k} \\ &= e^{j2\pi\frac{f_s\tau}{\beta T_u}k} e^{-j2\pi f_s\delta} e^{-\frac{j2\pi\delta}{T_u}k} \end{aligned} \tag{9}$$

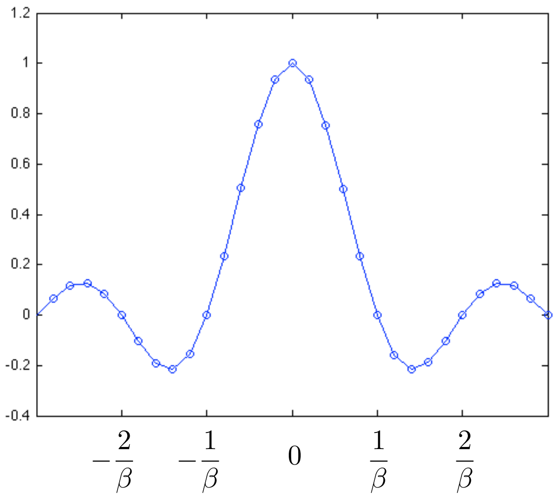

Fig. 1. Coherence of the Sensing matrix for single chirp with oversampled range. The columns are labeled in time samples with spacing $\frac{1}{c\beta}$, with $c = 5$.

The first term in (9) is a normalized frequency shift of magnitude $\overline{f}_s = \mod(\frac{-f_s\tau}{\beta T_u}, 1)$, circularly shifting the columns of the sensing matrix. The second term is a range dependent phase shift factor. Therefore, the column A_i of the sensing matrix for a radar sensor with the transmit waveform in (8) will be:

$$A_i(f_s) = e^{-j\frac{2\pi f_s}{c\beta}i} \begin{bmatrix} 1 & e^{j2\pi(\overline{f}_s + \frac{i}{cN})} & \dots & e^{j2\pi(N-1)(\overline{f}_s + \frac{i}{cN})} \end{bmatrix}^T$$

where the index i refers to the discretized range $\delta(i) = \frac{i}{c\beta}$ as before. The resulting sensing matrix $A(f_s)$ is a circularly shifted oversampled DFT matrix modulated by a phase ramp:

$$A(f_s) = [A_1(f_s) \quad A_2(f_s) \quad \cdots \quad A_{cN}(f_s)].$$

Now we consider the multi-frequency chirp waveform introduced in [2], [3]. The illumination waveform is obtained by superposition of chirps of equal chirp rate β/τ and with varying initial phases and frequency offsets:

$$\phi(t) = \sum_k e^{j\phi_k} e^{j2\pi f_k t} e^{j\pi\frac{\beta}{\tau}t^2}. \tag{10}$$

The sensing matrix A_c for the radar with compressive illumination on transmit and stretch processing on receive is superposition of circularly shifted oversampled DFT matrices each modulated with a phase ramp of rate $f_k/(c\beta)$:

$$A_c = \sum_k e^{j\phi_k} A(f_k).$$

To study the coherence of the columns of A we start with the limiting case of $K = cN$ LFM waveforms each separated by a frequency offset of $k\beta/N$:

$$\phi(t) = \sum_{k=0}^{cM-1} e^{j\phi_k} e^{j\frac{2\pi\beta k}{N}t} e^{j\pi\frac{\beta}{\tau}t^2}. \tag{11}$$

For this case the mutual coherence between the i and j'th column of the sensing matrix A_c is

$$\mu_{i,j} = \frac{1}{N}\sum_{k=0}^{K}\exp(\frac{2\pi k}{cN}|i-j|)D_N(\frac{2\pi}{cN}|i-j|) + \frac{1}{N}\sum_{k=0}^{K-1}\sum_{l=0}^{K-2} D_N(\frac{2\pi}{cN}|i-j+m_k-m_l|)e^{j(\phi_k-\phi l)} \tag{12}$$

where $m_k = c \mod(pk, N)$ represents the discrete frequency shift after aliasing with $p = \tau/T_u$ denoting the stretch factor. Second term is random sum of zero-mean complex numbers with finite variance and scales as the coherence of randomized sensing matrices. The first term is equal to $D_{cN}(2\pi/cN|i-j|)D_N(2\pi/cN|i-j|)/(cN^2)$ and eliminates the bias term that leads to the large coherence in the case of the single chirp waveform. Figure 2 shows the two components for the first term. We observe that by expanding the bandwidth of the transmitted waveform by a factor of c and uniform sampling of the start frequencies f_k over the bandwidth $[0, c\beta]$ we achieve low coherence between the columns. In the next section, through numerical simulations we

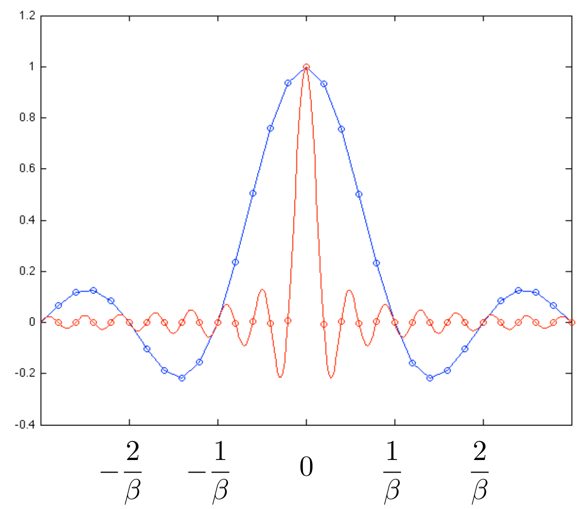

Fig. 2. Coherence of the Sensing matrix for single chirp with oversampled range. The columns are labeled in time samples with spacing $\frac{1}{c\beta}$, with $c = 5$.

show that low coherence can be achieved using a multi-frequency chirp waveform with much fewer components than the limiting case of cN chirps discussed above.

III. Numerical Study with Small Number of Chirp Components

As shown in the previous section, incorporating additional frequency shifted chirps in transmit minimizes the inter-column coherence of the sensing matrix after the stretch processing with a single chirp on receive. Eliminating large entries of the inter-column coherence is essential for stable recovery of a sparse signal from its noisy projections. The previous section provided a theoretical analysis based on large number of chirp components. Here we resort to numerical simulations to study the inter-column coherence of compressively illuminated stretch processor when few subcarriers are employed. In particular we consider multi-frequency linear FM signals with 750 MHz total bandwidth and 10 μsecond duration, composed of K subcarriers each with 50 MHz bandwidth. The center frequencies and complex phases of the subcarriers are randomly selected at each simulation run. The wideband received waveform is then dechirped using a *single* stretch processor with a single reference chirp of 50 MHz bandwidth and sampled at a rate of 5 Msample/sec of complex I/Q samples. Figure 3 shows the empirical histogram for of the inter-column coherences for multi-frequency chirp transmit waveform with $K = 1, 7, 15$ subcarriers. We observe that the percentage of large entries is significantly higher for the traditional single carrier waveform suggesting a higher percentage of target realizations will result in poor recovery with a standard LFM radar with $K = 1$ carrier.

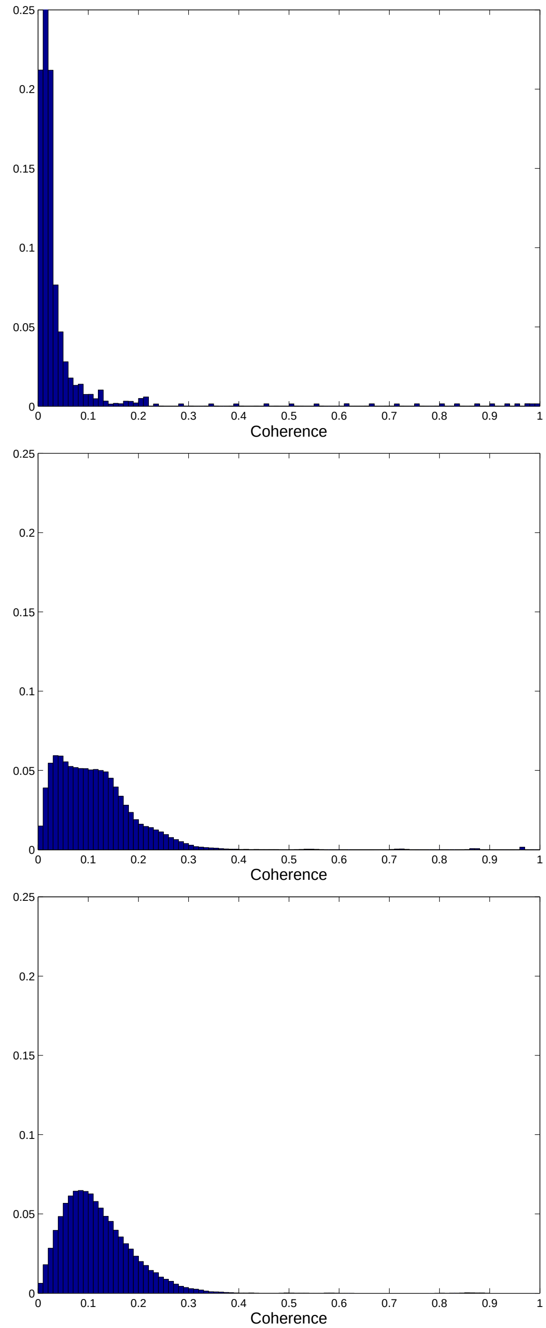

Fig. 3. Empirical histogram of the intercolumn coherences of multi-frequency chirp transmit waveforms with $M = 1, 7, 15$ subcarriers

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